Logic as a Vector System

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Abstract

This paper describes a diagrammatic system of propositional logic in which propositions are taken to be vectors or displacements in a ‘logical space’.

Keywords: Vectors, diagrammatic logic, logical space.

1 Introduction

In a letter written to Christian Huygens in 1679, which contains the earliest development of the theory of vectors [6, p. 3 ff]. Leibniz claims for his new ‘analysis of position’ or *analysis situs* that, ‘This new characteristic, which follows the visual figure, cannot fail to give the solution, the construction and the geometric demonstration all at the same time, that is, through determined procedure’ [9, p. 250]. That the same is true of the visual representation of logic functions as vectors is suggested by an observation of Wittgenstein’s in the *Tractatus Logico-Philosophicus*. At 4.461 he writes, in a geometrical simile, following the observation ‘Tautologies and contradictions lack sense’, that they are ‘Like a point from which two arrows go out in opposite directions to one another’.[1]

In this paper we take our cue from Leibniz and Wittgenstein. We develop a logical system in which propositions are vectors in logical space. The logical ‘force’ of the proposition is represented by characteristics of the vector corresponding to it. The resulting system has an easy graphical implementation, at least in cases involving few variables. Cases with more than three variables are more difficult graphically (as each variable requires a new dimension) but can be easily dealt with using the already well-developed mathematics of vectors.

Of course the idea that logic might be visualized or represented geometrically is not at all new. Venn diagrams are commonly used in beginning logic instruction [8] and can themselves be seen as improvements on Euler diagrams. Lewis Carroll presents diagrams that he sees as further improvements [5]. There have also been attempts to understand the logic of diagrams generally. The work of Jon Barwise and John Etchemendy [3, 2] is important here. Finally, many studies have attempted to spell out the logical workings of already existent diagrammatic methods, for example, Sun-Joo Shin’s work on the logic of several diagrammatic systems [12, 13, 14].

More closely related to our work are systems such as arrow logic[2] and digraphs [7]. However, these systems do not quantify the magnitude or direction of the arrows they use. As a result, neither can be seen as a vector system.

We do not wish, however, to suggest that our work occurs in a vacuum, nor that we are the first to envision a relation between vectors and logic. Widdows and Peters [16] have argued that logical operations within certain kinds of vector spaces provide a perspicuous and computationally efficient way of modelling word meaning within the context of document retrieval operations. Vectors, in

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[1] ‘Wie der Punkt, von dem zwei Pfeile in entgegengesetzer Richtung auseinandergehen.’ [17] p. 68 and p. 69.

[2] The best introduction to the subject is [15]

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doi:10.1093/logcom/exi040

their system, are models of word meaning. Mizraji [10] has developed a vector-based logic in which truth-values are vectors and logical operators are matrix operations on the space. More closely related to the present work is von Neumann's quantum logic [4]. Von Neumann models propositions as sets of vectors in Hilbert Space. However, in contrast to the present work, von Neumann's logic is non-classical. More recently, Aiello and van Benthem [1] have suggested that mathematical morphology provides a model for linear logic. On such a model, formulas are interpreted as arbitrary subsets of vector space. However, the logic is again not classical.

Given the history, it would be surprising if anyone were to deny that there was a relation between vectors and logic. Nonetheless no one has, so far as we know, proposed modelling propositions as vectors while retaining classical logic. We hope that by the end of this paper the reader will agree that doing so is both interesting and fruitful.

2 Logical space

Let there be a space in which the co-ordinates, taken from the origin $\boldsymbol{O}$, are propositional addresses, in the sense that the propositional address $(\boldsymbol{p})$ is $(\mathbf{1},\mathbf{0})$, and the propositional address $(\boldsymbol{q})$ is $(\mathbf{0},\mathbf{1})$. $(\mathbf{1},\mathbf{1})$ will be the propositional address $(\boldsymbol{p},\boldsymbol{q})$, defined, like the other two addresses, by distance and direction from $\boldsymbol{O}$. The proposition p can now be represented as a directed line segment or vector travelling along the abscissa (the p-axis) from $\boldsymbol{O}$ to $(\boldsymbol{p})$, and the proposition q represented as a directed line segment travelling up the ordinate or q-axis from $\boldsymbol{O}$ to $(\boldsymbol{q})$ (see Figure 1).

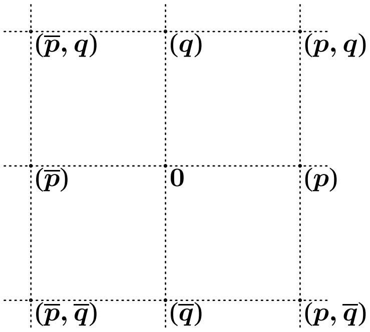
$(\overline{\boldsymbol{p}},\boldsymbol{q})$
$(\boldsymbol{q})$
$(\boldsymbol{p},\boldsymbol{q})$
$(\overline{\boldsymbol{p}})$
$\mathbf{0}$
$(\boldsymbol{p})$
$(\overline{\boldsymbol{p}},\overline{\boldsymbol{q}})$
$(\overline{\boldsymbol{q}})$
$(\boldsymbol{p},\overline{\boldsymbol{q}})$

FIGURE 1. A space for propositions

The vector $\boldsymbol{O} \rightarrowtail (\boldsymbol{p},\boldsymbol{q})$ is to be interpreted as expressing $p \vee q$, and similarly for other vectors proceeding from the origin[3]. In accordance with this principle, the vector from $\boldsymbol{O}$ to $(\overline{\boldsymbol{p}},\boldsymbol{q})$ or $(\mathbf{-1},\mathbf{1})$ expresses $\overline{p} \vee q$, and similarly for $p \vee \overline{q}$ and $\overline{p} \vee \overline{q}$.

So we are able to express all those propositions which use only $\vee$ and the literals.[4] Alternations of literals are expressed by vectors which begin at $\boldsymbol{O}$. Conjunctions may be expressed by vectors which end at $\boldsymbol{O}$[5]. Thus $(\boldsymbol{p},\boldsymbol{q}) \rightarrowtail \boldsymbol{O}$ expresses pq. Similarly, we read $(\boldsymbol{p},\overline{\boldsymbol{q}}) \rightarrowtail \boldsymbol{O}$ as $p\overline{q}$, not as

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[3] Our strategy here, and throughout the paper, is to draw a parallel between vectors in our system and the propositions of propositional logic. Our results thus depend on well-known results about propositional logic. Bold face notation indicates objects in the vector space while non-bold represents objects in propositional logic. In technical results we use '$[\![\boldsymbol{V}]\!]$' to mean 'the proposition expressed by $\boldsymbol{V}$'.

[4] There is an exception to this claim. Those propositions which use both a literal and its negation, e.g. $p \vee \overline{p}$, are not expressible as no point contains both a literal and its negation. These propositions will be accommodated in 2.1.

[5] Although the choice of having outward vectors express conjunctions while inward vectors express alternations may seem arbitrary, it is not. However, it depends on materials from later sections.

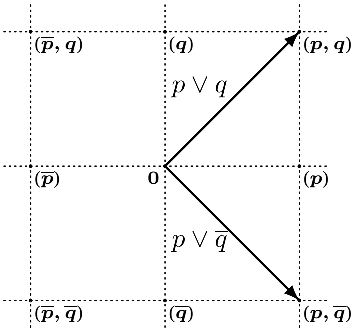

FIGURE 2. The propositions $p \vee q$ and $p \vee \overline{q}$ as vectors

$p \vee \overline{q}$. So when a vector travelling in the $(\overline{p}, q)$ direction is directed into the origin, it is the *negation* of the vector travelling *from* the origin in the same direction.

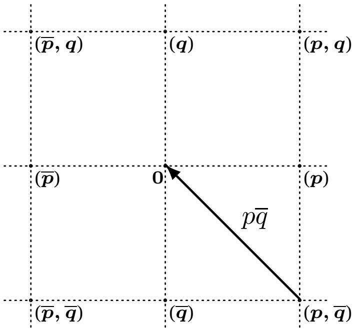

FIGURE 3. $p\overline{q}$

In general, if one end of a vector is at $\boldsymbol{O}$, then reversing the direction of the vector yields the dual of the proposition expressed by the original vector. It is important to note that since p is self-dual, $\boldsymbol{O} \rightarrowtail (\boldsymbol{p})$ and $(\boldsymbol{p}) \rightarrowtail \boldsymbol{O}$ both express p.

2.1 ANF and CNF

We can already express, with a single vector, propositions that are pure alternations or pure conjunctions of literals. However, this falls significantly short of yielding the full expressive power of propositional logic. In order to achieve expressive completeness we need to introduce the notion of a *vector system*. In a vector system multiple vectors jointly express a single proposition. A system of vectors travelling *from* the origin represents a proposition in CNF form: the conjunction of the alternations expressed by the individual vectors of the system. A system of vectors *to* $\boldsymbol{O}$ represents a proposition in ANF form: the alternation of the conjunctions expressed by the individual vectors of the system. Vector systems may thus express any proposition which is in either ANF or CNF form, a fact which will allow us to prove that vector logic is expressively complete (Theorem 2.2). Vector systems must be uniform in the sense that the vectors of the system must either all begin at $\boldsymbol{O}$ or end at $\boldsymbol{O}$; we do not allow mixed systems. We say that a vector has ANF form if its endpoint is $\boldsymbol{O}$ (so that it could be part of a system expressing an ANF propositions) and that it has CNF form if it begins at $\boldsymbol{O}$ (and thus could be part of a system expressing a CNF proposition.)

Given this notion of vector system, we can represent propositions that could not be represented by single vectors. For example, we can now represent $p \vee \overline{p}$, a tautology, as a system of two vectors which travel *to* the origin. The vector $\mathbf{0} \longrightarrow (\boldsymbol{p})$ expresses p while $(\overline{\boldsymbol{p}}) \rightarrowtail \mathbf{0}$ expresses $\overline{p}$. As a system, the two vectors express the alternation of the propositions they would express individually, i.e. $p \vee \overline{p}$. (Figure 4)

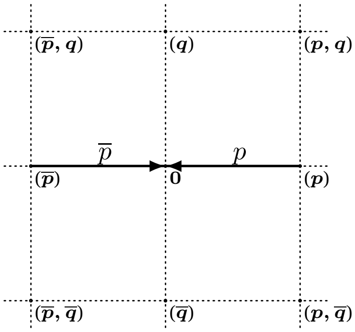
$(\overline{\boldsymbol{p}}, \boldsymbol{q})$
$(\boldsymbol{q})$
$(\boldsymbol{p}, \boldsymbol{q})$
$\overline{p}$
p
$(\overline{\boldsymbol{p}})$
$\mathbf{0}$
$(\boldsymbol{p})$
$(\overline{\boldsymbol{p}}, \overline{\boldsymbol{q}})$
$(\overline{\boldsymbol{q}})$
$(\boldsymbol{p}, \overline{\boldsymbol{q}})$

FIGURE 4. $p \vee \overline{p}$

An interesting example of a conjunction is the contradiction, shown in Figure 5. If the two vectors representing p and $\overline{p}$ are made to travel to $(\boldsymbol{p})$ and $(\overline{\boldsymbol{p}})$ *from* $\boldsymbol{O}$, the system expresses their conjunction.

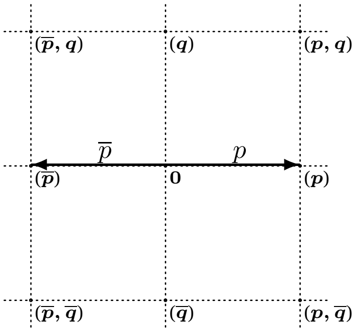
$(\overline{\boldsymbol{p}}, \boldsymbol{q})$
$(\boldsymbol{q})$
$(\boldsymbol{p}, \boldsymbol{q})$
$\overline{p}$
p
$(\overline{\boldsymbol{p}})$
$\mathbf{0}$
$(\boldsymbol{p})$
$(\overline{\boldsymbol{p}}, \overline{\boldsymbol{q}})$
$(\overline{\boldsymbol{q}})$
$(\boldsymbol{p}, \overline{\boldsymbol{q}})$

FIGURE 5. $p\overline{p}$

Some propositions, such as $p \vee q$, can be thought of as degenerate and can be read as having indifferently ANF or CNF. Is $p \vee q$ a single-conjunct in CNF, for example in a ‘series’ such as $(p \vee q)(p \vee q)\ldots$, or is it a schema such as $pp \ldots \vee qq \ldots$, to which it is equivalent, in ANF? This ‘double aspect’ of $p \vee q$ is represented in vector form by the fact that the vector $\boldsymbol{O} \rightarrowtail (\boldsymbol{p}, \boldsymbol{q})$ and the vector *system* consisting of the vector $(\boldsymbol{q}) \rightarrowtail \boldsymbol{O}$ and the vector $(\boldsymbol{p}) \rightarrowtail \boldsymbol{O}$ both express the same proposition. This is shown in Figure 6 with thick lines representing the vector systems and thin lines representing the single vectors. Similarly, pq has two forms, the vector $(\boldsymbol{p}, \boldsymbol{q}) \rightarrowtail \boldsymbol{O}$, and the CNF vector system consisting of $\boldsymbol{O} \rightarrowtail (\boldsymbol{p})$ and $\boldsymbol{O} \rightarrowtail (\boldsymbol{q})$.

Theorem 2.1 follows easily from the definitions of ANF and CNF vectors and systems.

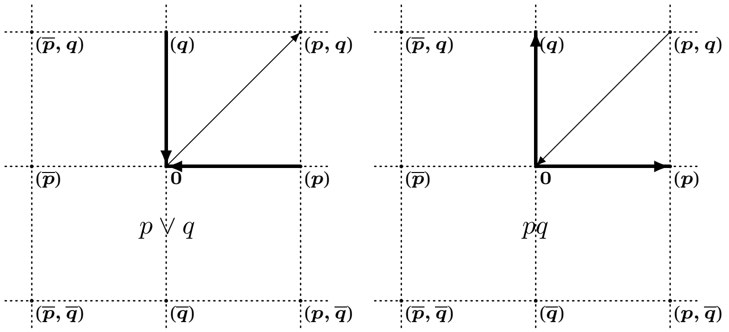

FIGURE 6. Two forms of $p \vee q$ and pq

THEOREM 2.1 (Conversion)
Let $\boldsymbol{V}$ be a vector in either ANF or CNF form whose non-$\mathbf{0}$ vertex is $\boldsymbol{x_0, x_1, \ldots x_n}$, then $[\![\boldsymbol{V}]\!] = [\![\boldsymbol{X_0, X_1, \ldots X_n}]\!]$ where each $\boldsymbol{X_i}$ is a vector of opposite form from $\boldsymbol{V}$ and whose non-$\mathbf{0}$ vertex is $\boldsymbol{x_i}$.

We are now in a position to prove that the system thus described is expressively complete with respect to propositional logic. We use the notation '$[\![\boldsymbol{V}]\!]$' to mean 'the proposition expressed by $\boldsymbol{V}$'. Where $\boldsymbol{V}$ is a list of vectors, the notation picks out the proposition expressed by the vector system comprised of the listed vectors.

THEOREM 2.2 (Expressive completeness)
Let Φ be a formula of classical propositional logic, then there is a system of vectors $\boldsymbol{V_0, V_1, \ldots V_n}$ that expresses Φ, (*i.e.* such that $[\![\boldsymbol{V_0, V_1, \ldots V_n}]\!]$ is Φ).

PROOF. This theorem follows easily from that fact that every formula of propositional logic is equivalent to a formula in alternational normal form and the fact that any conjunction of literals is expressible by a vector starting from the point containing exactly those literals and ending at $\mathbf{0}$.

Given a proposition Φ, we convert it to an equivalent alternational normal form proposition $\Phi^{\vee}$. By definition, $\Phi^{\vee}$ is $\Psi_1 \vee \Psi_2 \vee \ldots \vee \Psi_n$ where each Ψ_i is a conjunction of literals. Now let the literals in Ψ_i be $\Psi_i^1, \Psi_i^2, \ldots, \Psi_i^k$. Ψ_i may thus be expressed by the vector $V_{\Psi_i} =(\boldsymbol{\Psi_i^1, \Psi_i^2, \ldots, \Psi_i^k}) \rightarrowtail \mathbf{0}$. So $\Phi^{\vee}$ may be expressed by the system of vectors $V_{\Psi_1}, V_{\Psi_2}, \ldots, V_{\Psi_n}$.

A similar proof can be had based upon conjunctive normal form.[6] ∎

2.2 Translation

One important feature of vectors is their equivalence under translation. In this section we see that logical vectors are also equivalent under translation with one restriction.

If we translate the vector $\boldsymbol{O} \rightarrowtail (\overline{\boldsymbol{p}}, \boldsymbol{q})$ one unit in the $(\boldsymbol{p})$ direction, we find $(\boldsymbol{p}) \rightarrowtail (\boldsymbol{q})$ which we can take as expressing $p \rightarrow q$, as shown in Figure 7. But these two vectors express *equivalent* propositions. Either of them can also be translated into the third quadrant, where we find $(\overline{\boldsymbol{q}}) \rightarrowtail (\overline{\boldsymbol{p}})$, which is also equivalent to $\overline{p} \vee q$ and to $p \rightarrow q$. These translations define implication and transposition, respectively, and we arrive at the proposition that the translation of a vector gives an equivalence, except in cases when one of the vectors terminates at the origin, e.g. as with the translation of $(\boldsymbol{p}) \rightarrowtail (\boldsymbol{q})$ to $(\boldsymbol{p}, \overline{\boldsymbol{q}}) \rightarrowtail \boldsymbol{O}$.

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[6] It might seem that that this proof really requires a full statement of the semantics of vectors. This would be correct insofar as a claim of expressive completeness is a claim about the relationship between the syntax and semantics of a logical system. However, our aim is just to show that our vector system is a useful new syntax for propositional logic. Thus in this and other theorems we need only show a relation between vectors and the propositions or sentences of propositional logic. Our claim of expressive completeness is just that for any sentence in the standard syntax of propositional logic, there is a vector in our system which expresses that sentence. Given, then, that propositional logic can express every Boolean truth-function, so can our vector system.

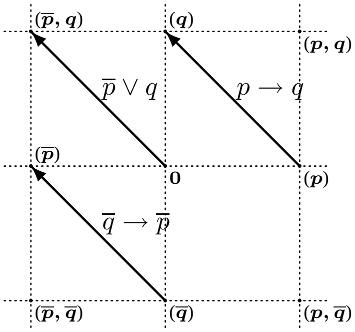
$(\overline{\boldsymbol{p}}, \boldsymbol{q})$
$(\boldsymbol{q})$
$(\boldsymbol{p}, \boldsymbol{q})$
$\overline{p} \vee q$
$p \rightarrow q$
$(\overline{\boldsymbol{p}})$
$\boldsymbol{0}$
$(\boldsymbol{p})$
$\overline{q} \rightarrow \overline{p}$
$(\overline{\boldsymbol{p}}, \overline{\boldsymbol{q}})$
$(\overline{\boldsymbol{q}})$
$(\boldsymbol{p}, \overline{\boldsymbol{q}})$

FIGURE 7. Translations of $\boldsymbol{p} \rightarrowtail \boldsymbol{q}$

We are now in a position to understand why outward vectors should express alternations and inward vectors should express conjunctions. We want translation to hold as widely as possible, compatible with expressive completeness. Expressive completeness require that some vectors express conjunctions while others express disjunctions. It is very natural to read the vector $(\boldsymbol{p}) \rightarrowtail (\boldsymbol{q})$ as expressing $p \rightarrow q$ (which is equivalent to $\overline{p} \vee q$). $(\overline{\boldsymbol{q}}) \rightarrowtail (\overline{\boldsymbol{p}})$ is naturally taken to express $\overline{q} \rightarrow \overline{p}$ which is also equivalent to $\overline{p} \vee q$. So of the two vectors $\boldsymbol{O} \rightarrowtail (\overline{\boldsymbol{p}}, \boldsymbol{q})$, $(\boldsymbol{p}, \overline{\boldsymbol{q}}) \rightarrow \boldsymbol{O}$, one should express $\overline{p} \vee q$. Clearly $\boldsymbol{O} \rightarrowtail (\overline{\boldsymbol{p}}, \boldsymbol{q})$ is the more perspicuous choice, especially as this leaves the remaining vector free to express $p\overline{q}$ perspicuously. Thus there is a natural limit to the representation of logic vectors by displacement matrices, as the matrices cannot express the restriction placed on the bound vector, or its relation to the origin.

The starting point of each vector, together with the direction, gives the end-point, and so we have a cancelation technique in which the starting point $\boldsymbol{O}$ is the cancelation of no literal, and the endpoint $\boldsymbol{O}$ is the cancelation of all the literals.

In general, if $\boldsymbol{\mathcal{A}}$ and $\boldsymbol{\mathcal{B}}$ are points in a logical space as described above, and $\boldsymbol{\mathcal{B}} \neq \boldsymbol{O}$, then the vector $\boldsymbol{\mathcal{A}} \rightarrowtail \boldsymbol{\mathcal{B}}$ expresses $\bigwedge \alpha \rightarrow \bigvee \beta$ where α is the set of literals in $\boldsymbol{\mathcal{A}}$ and β is the set of literals in $\boldsymbol{\mathcal{B}}$. If $\boldsymbol{\mathcal{B}} = \boldsymbol{O}$, then $\boldsymbol{\mathcal{A}} \rightarrowtail \boldsymbol{\mathcal{B}}$ expresses $\bigwedge \alpha$.

Call a vector *minimal* if no propositional letter occurs in both of its vertices. We can think of minimality as embodying two weaker requirements. The first is that a vector should not have a length[7] greater than 1. The second is that the vector be embedded in the smallest dimensional space possible. Thus a vector which does not change in a given dimension is minimal only if the initial and final values in that dimension are 0. We will generally assume that vectors are minimal unless otherwise stated.

THEOREM 2.3 (Translation)
If $\boldsymbol{\mathcal{C}} \rightarrowtail \boldsymbol{\mathcal{D}}$ is a translation of $\boldsymbol{\mathcal{A}} \rightarrowtail \boldsymbol{\mathcal{B}}$, then $[\![\boldsymbol{\mathcal{A}} \rightarrowtail \boldsymbol{\mathcal{B}}]\!] = [\![\boldsymbol{\mathcal{C}} \rightarrowtail \boldsymbol{\mathcal{D}}]\!]$, provided that $\boldsymbol{\mathcal{B}}, \boldsymbol{\mathcal{D}} \neq \boldsymbol{O}$ and both vectors are minimal.

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[7] Since we are dealing with a Boolean space, we take the length of a vector to be its maximum delta in any dimension. Thus $\boldsymbol{O} \rightarrowtail (\boldsymbol{p}, \boldsymbol{q})$ has a length of 1 rather than $\sqrt{2}$.

PROOF. $[\![\boldsymbol{\mathcal{A}} \rightarrowtail \boldsymbol{\mathcal{B}}]\!] = \bigwedge \alpha \rightarrow \bigvee \beta$ and hence $= \neg \bigwedge \alpha \vee \bigvee \beta$. By de Morgan's theorem we get $\bigvee \alpha^* \vee \bigvee \beta$ where α^* is the set of negations of members of α. Now let K be an arbitrary literal in α^* or β containing the propositional letter k. By minimality k cannot occur more than once in α and β. Further, K's sign is determined by the direction of change of $\mathcal{A} \rightarrowtail \mathcal{B}$ in the k dimension. If $\mathcal{A} \rightarrowtail \mathcal{B}$ increases in the k dimension, then K will be k; if $\mathcal{A} \rightarrowtail \mathcal{B}$ decreases in the k dimension, then K is $\overline{k}$. This is because $\mathcal{A} \rightarrowtail \mathcal{B}$ can increase in the k dimension only if its value in that dimension is -1 at $\boldsymbol{\mathcal{A}}$ or 1 at $\boldsymbol{\mathcal{B}}$. These both yield k in the disjunctive form of $[\![\boldsymbol{\mathcal{A}} \rightarrowtail \boldsymbol{\mathcal{B}}]\!]$ because the values in $\boldsymbol{\mathcal{A}}$ are inverted by the application of de Morgan's theorem. A similar argument shows that if $\mathcal{A} \rightarrowtail \mathcal{B}$ is decreasing in the k dimension, then K is negative. Since $\boldsymbol{\mathcal{C}} \rightarrowtail \boldsymbol{\mathcal{D}}$ is a translation of $\boldsymbol{\mathcal{A}} \rightarrowtail \boldsymbol{\mathcal{B}}$, either both vectors increase in the k dimension or decrease in it. This means that K will be the same in the disjunctive form of the interpretation of both vectors. But since K was an arbitrary literal in $[\![\boldsymbol{\mathcal{A}} \rightarrowtail \boldsymbol{\mathcal{B}}]\!]$, $[\![\boldsymbol{\mathcal{C}} \rightarrowtail \boldsymbol{\mathcal{D}}]\!] = \overline{\bigwedge \alpha} \vee \bigvee \beta$. ■

Translating vectors through the origin expresses de Morgan's theorems (Figure 8). If, for example, $\boldsymbol{O} \rightarrowtail (\boldsymbol{p}, \boldsymbol{q})$ is slid through $\boldsymbol{O}$, it becomes its negation. The sliding of a vector through $\boldsymbol{O}$, then, is the operation of negation.

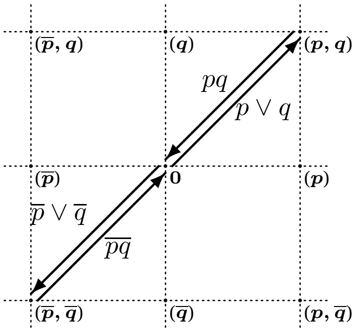
$(\overline{\boldsymbol{p}}, \boldsymbol{q})$
$(\boldsymbol{q})$
$(\boldsymbol{p}, \boldsymbol{q})$
pq
$p \vee q$
$(\overline{\boldsymbol{p}})$
$\mathbf{0}$
$(\boldsymbol{p})$
$\overline{p} \vee \overline{q}$
$\overline{pq}$
$(\overline{\boldsymbol{p}}, \overline{\boldsymbol{q}})$
$(\overline{\boldsymbol{q}})$
$(\boldsymbol{p}, \overline{\boldsymbol{q}})$

FIGURE 8. de Morgan's Theorems

The translations of $\boldsymbol{p} \rightarrowtail \boldsymbol{q}$ reveal the relationship between the operation of negating, an operation that we call N^* which negates each literal in a proposition,[8] and the relation of duality Δ. Sliding a vector through $\boldsymbol{O}$ has the effect of negating it, or of transforming it into its N^* dual. The following relationships are noteworthy:

> The *negation* of a proposition α is the dual of the N^* of α, or $\overline{\alpha} = (\Delta(N^*(\alpha)))$.
>
> The *dual* Δ of a proposition α is the N^* of the negation of α, or $\Delta(\alpha) = (N^*(\overline{\alpha}))$.
>
> The N^* of a proposition α is the negation of the dual of α, or $N^*(\alpha) = \overline{\Delta(\alpha)}$.

To illustrate these relationships, let α be $p \vee q$. The *negation* of $p \vee q$ is $\overline{p}\,\overline{q}$. The *dual* of $p \vee q$ is pq, of which the N^* is $\overline{p}\,\overline{q}$. The N^* of $p \vee q$ is $\overline{p} \vee \overline{q}$, of which the negation is pq. The N^* of $p \vee q$ is $\overline{p} \vee \overline{q}$. The negation of $p \vee q$ is $\overline{p}\,\overline{q}$, whose dual is $\overline{p} \vee \overline{q}$. The order of operations is immaterial, so that for example the dual of the negation of α is also the negation of the dual of α. The three principles given above work nicely when α is a literal such as p, as p is self-dual.

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[8] Operator N in Wittgenstein's *Tractatus* is a generalization of negation to many places, so that for example N(pqr) is $\overline{p}\,\overline{q}\,\overline{r}$. We can imagine a parallel operation (Russel Wahl has suggest calling it S, for 'Sheffer') which transforms an alternation such as $p \vee q \vee r$ into $\overline{p} \vee \overline{q} \vee \overline{r}$. '$N^*$' then selects N *or* S, as appropriate.

THEOREM 2.4
In the vector representation, if $\boldsymbol{V}$ is $\boldsymbol{O} \rightarrowtail \boldsymbol{B}$, and $\boldsymbol{B'}$ has the opposite sign of $\boldsymbol{B}$ in every dimension,

1. the negation $\overline{\boldsymbol{V}}$ of $\boldsymbol{V}$ is the sliding of $\boldsymbol{V}$ through $\boldsymbol{O}$ (i.e. $\boldsymbol{B'} \rightarrowtail \boldsymbol{O}$);
2. the dual $\boldsymbol{V}^{\triangle}$ of $\boldsymbol{V}$ is the reverse vector of $\boldsymbol{V}$ at $\boldsymbol{O}$ (i.e. $\boldsymbol{B} \rightarrowtail \boldsymbol{O}$);
3. the N^* $\boldsymbol{V}^*$ of $\boldsymbol{V}$ is the reflection of $\boldsymbol{V}$ in $\boldsymbol{O}$ (i.e. $\boldsymbol{O} \rightarrowtail \boldsymbol{B'}$).

PROOF. These all follow easily from the fact that vectors beginning at $\boldsymbol{O}$ express the disjunction of the literals at their endpoint, while vectors ending at $\boldsymbol{O}$ express the conjunction of the literals at their starting point. ■

From Theorems 2.3 and 2.4 it is easy to describe the negation, dual, and N^* of any minimal vector. It is also worth noting at this point that since p is self-dual, the vectors from $\boldsymbol{O}$ to $(\boldsymbol{p})$ and from $(\boldsymbol{p})$ to $\boldsymbol{O}$ are equivalent. They both express p, a fact which can be important in some proofs.

3 Logical principles

Translation and conversion are principles of equivalence in vector logic. There are two rule of inference, a restricted form of vector addition and weakening. Together these rules capture a surprising amount of logic.

3.1 Addition

Traditionally there are two methods of illustrating vector addition — the triangle method and the parallelogram method. For numerical vectors, these two methods are equivalent; however, they are not equivalent for logic vectors. The reason is that logical vectors cannot all be translated freely. Thus vector addition in logical space is only valid in a restricted form. Interestingly, the restrictions are different depending on whether the vectors to be added are in ANF or CNF.

The first restriction, common to both forms, is that vector addition is only allowed when the addends are either all in CNF or all in ANF. Of course, since we are dealing with a Boolean space, addition is only defined when the resultant vector would start and end at an integer valued point. ANF vectors may only be added with the parallelogram method. CNF vectors may only be added by the triangle method, and then only when the two addends meet on an axis.

The difference in restrictions on ANF versus CNF addition stems from the fact that in our system whether a vector is bound or free is determined by the position and direction of the vector, and the fact that we interpret bound and free vectors differently. Thus if a free vector is translated so that its endpoint is at $\boldsymbol{O}$, it becomes bound and receives a new interpretation which is generally not equivalent to the old one. Bound numerical vectors may only be added via the triangle method; our logical vectors are the same in this respect. The parallelogram method, however, involves a sort of virtual translation that when applied to CNF vectors can violate the restriction regarding addends meeting on an axis and may even involve a virtual change of the vector to ANF, or bound, form.

3.1.1 ANF addition

Because ANF vectors are bound vectors that share a common endpoint, it is impossible to form chains of vectors with them. Hence it is impossible to use the triangle method of addition. However, the parallelogram method does not require the translation of vectors and it is valid for ANF systems.

The following theorem holds for adding individual bound vectors. It is important to remember that when ANF vectors occur as systems, they cannot be added in this fashion. Instead they should be converted to their CNF form and then added if possible. If this restriction is not observed, invalid inferences can occur. For example, from $\boldsymbol{O} \rightarrowtail (\boldsymbol{p}, \boldsymbol{q})$ we can obtain the system $(\boldsymbol{p}) \rightarrowtail \boldsymbol{O}$, $(\boldsymbol{q}) \rightarrowtail \boldsymbol{O}$ by conversion. If we were to perform addition on the components of this system, we would obtain $(\boldsymbol{p}, \boldsymbol{q}) \rightarrowtail \boldsymbol{O}$. This would amount to inferring pq from $p \vee q$, an invalid inference. However if $(\boldsymbol{p}) \rightarrowtail \boldsymbol{O}$ and $(\boldsymbol{q}) \rightarrowtail \boldsymbol{O}$ are given as individual premisses, then addition may be performed keeping in mind that any values greater than 1 or less than -1 are reduced to 1 and -1 respectively.

THEOREM 3.1 (ANF addition)
Let $\boldsymbol{V}$ and $\boldsymbol{W}$ be ANF vectors. Then $\{[\![\boldsymbol{V}]\!], [\![\boldsymbol{W}]\!]\} \vDash [\![\boldsymbol{V} + \boldsymbol{W}]\!]$.

PROOF. Let $\boldsymbol{v_1}, \boldsymbol{v_2}, \ldots \boldsymbol{v_n}$ and $\boldsymbol{w_1}, \boldsymbol{w_2}, \ldots \boldsymbol{w_n}$ be the starting vertices of $\boldsymbol{V}$ and $\boldsymbol{W}$ where n is the smallest number of dimensions capable of embedding both $\boldsymbol{V}$ and $\boldsymbol{W}$, and let $\boldsymbol{U}{=}\boldsymbol{V} + \boldsymbol{W}{=}(\boldsymbol{u_1}, \boldsymbol{u_2}, \ldots \boldsymbol{u_n}) \rightarrowtail \boldsymbol{0}$.

If for some n, $\boldsymbol{v_n}, \boldsymbol{w_n}$ have opposite signs, then $[\![\boldsymbol{V}]\!]$, $[\![\boldsymbol{W}]\!]$ are mutually inconsistent and any inference from them is trivially valid.

Otherwise, for each n, $\boldsymbol{u_n}{=}\boldsymbol{v_n}$ or $\boldsymbol{u_n}{=}\boldsymbol{w_n}$. But this means that $[\![\boldsymbol{U}]\!]$ is a conjunction of literals each of which was one of the conjuncts expressed by either $\boldsymbol{V}$ or $\boldsymbol{W}$. Thus $\{[\![\boldsymbol{V}]\!], [\![\boldsymbol{W}]\!]\} \vDash [\![\boldsymbol{U}]\!]$. ■

3.1.2 CNF addition

Addition of CNF vectors is valid only when using the triangle method. That is, it must be possible to translate the vectors so as to form a chain. Further the 'joints' of the chain are restricted in a certain way.

If $\boldsymbol{V}$ and $\boldsymbol{W}$ are vectors that share a common point, we call that point the *joint* of the two vectors. The dimensionality of a point is just the number of non-zero values at that point. Thus the origin is 0-dimensional, any point on an axis is 1-dimensional, etc.

THEOREM 3.2 (CNF Addition)
Let $\boldsymbol{V_0}$, $\boldsymbol{V_1}$, …, $\boldsymbol{V_n}$ be a chain of vectors with only 1-dimensional joints and let α and β be the starting and ending points of $\boldsymbol{V_0}$ and $\boldsymbol{V_n}$ respectively. $\{[\![\boldsymbol{V_0}]\!], [\![\boldsymbol{V_1}]\!], \ldots, [\![\boldsymbol{V_n}]\!]\} \vDash [\![\boldsymbol{\alpha} \rightarrowtail \boldsymbol{\beta}]\!]$.

PROOF. The theorem follows easily from the interpretation of vectors. Note that only $\boldsymbol{V_0}$ can start at $\boldsymbol{0}$.[9] The remaining vectors in the chain must each express a conditional whose antecedent is the disjunction of the literals at its starting point and whose consequent is the conjunction of the literals at its end point. The restriction to 1-dimensional joints guarantees that whenever two vectors are chained, the consequent of the proposition expressed by the first vector is equivalent to the antecedent of the proposition expressed by the second. The sum of the two vectors then follows as an instance of hypothetical syllogism, though it would not if we allowed higher dimensional joints. If there are further vectors in the chain, they can be added one by one to the new summed vector, as by construction this latter cannot violate the requirement of 1-dimensional joints. ■

3.2 Weakening

Free vectors may have their dimensionality increased and ANF vectors may have their dimensionality decreased. We define a *superpoint*, $\boldsymbol{\alpha}^{\geq}$, of $\boldsymbol{\alpha}$ as a point which agrees with all non-zero values in $\boldsymbol{\alpha}$. An equivalent formulation is that $\boldsymbol{\beta}$ is a superpoint of $\boldsymbol{\alpha}$ just in case the set of literals in $\boldsymbol{\beta}$ is a superset of the set of literals in $\boldsymbol{\alpha}$. *Subpoint* can be defined in the obvious way along with strict versions of both superpoint and subpoint.

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[9] If some other vector in the chain, say $\boldsymbol{V_i}$, started at $\boldsymbol{O}$, then $\boldsymbol{V_{i-1}}$ would would have to have ended at $\boldsymbol{O}$ and hence would not be in CNF form.

THEOREM 3.3 (Free vector weakening)
Let $\boldsymbol{\alpha} \rightarrowtail \boldsymbol{\beta}$ be a free vector. Then $\llbracket \boldsymbol{\alpha} \rightarrowtail \boldsymbol{\beta} \rrbracket \vDash \llbracket \boldsymbol{\alpha}^{\geq} \rightarrowtail \boldsymbol{\beta}^{\geq} \rrbracket$ for all superpoints of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$.

PROOF. This follows from the fact that $(p \wedge q) \rightarrow (r \vee s) \vDash (p \wedge q \wedge t) \rightarrow (r \vee s \vee u)$. ■

THEOREM 3.4 (ANF weakening)
Let $\boldsymbol{\alpha} \rightarrowtail \mathbf{0}$ be an ANF vector. Then $\llbracket \boldsymbol{\alpha} \rightarrowtail \mathbf{0} \rrbracket \vDash \llbracket \boldsymbol{\alpha}^{\leq} \rightarrowtail \mathbf{0} \rrbracket$ for any subpoint of $\boldsymbol{\alpha}$.

PROOF. This follows from the fact that $p \wedge q \wedge r \vDash p \wedge q$. ■

4 Examples

In this section we work through some examples of standard inferences, showing how they work in a vector system as described above. Our goal, for now, is to show how the system can be applied rather than to develop a proof of completeness. We will thus be giving examples of how a rule such as *Modus Ponens* might be used rather than presenting a proof that every instance of the rule can be captured in our system.

4.1 Modus Ponens

We can represent $p \rightarrow q$ in a very natural way as the vector from $(\boldsymbol{p})$ to $(\boldsymbol{q})$. Figure 9 shows *Modus Ponens*.

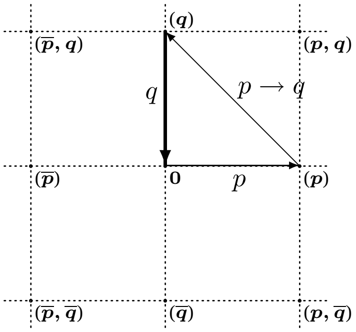
$(\boldsymbol{q})$
$(\overline{\boldsymbol{p}}, \boldsymbol{q})$
$(\boldsymbol{p}, \boldsymbol{q})$
q
$p \rightarrow q$
$(\overline{\boldsymbol{p}})$
$\mathbf{0}$
p
$(\boldsymbol{p})$
$(\overline{\boldsymbol{p}}, \overline{\boldsymbol{q}})$
$(\overline{\boldsymbol{q}})$
$(\boldsymbol{p}, \overline{\boldsymbol{q}})$

FIGURE 9. *Modus Ponens*

If the vectors are represented as displacements, then *Modus Ponens* is the following set of summed displacements.

		p	q
Premiss 1	$p \rightarrow q$	-1	1
Premiss 2	p	1	0
Conclusion	q	0	1

Modus Tollens is shown in Figure 10. In order to meet the constraint that addition use the triangle method, the first first premiss has been translated to the third quadrant. The sum of the displacements is:

		p	q
Premiss 1	$p \longrightarrow q$	-1	1
Premiss 2	$\overline{q}$	0	-1
Conclusion	$\overline{p}$	-1	0

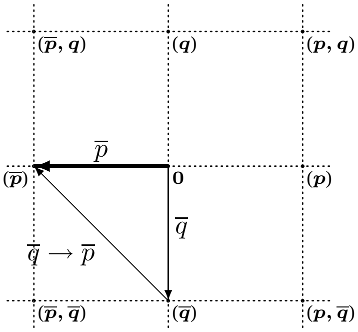

FIGURE 10. *Modus Tollens*

Exactly the same principles apply to the disjunctive syllogism, including the need to translate the first premiss to a different quadrant.

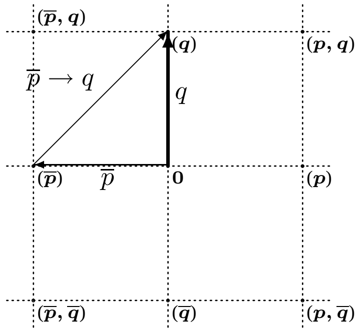

FIGURE 11. Disjunctive syllogism

The displacement matrices and their sum are:

		p	q
Premiss 1	$p \vee q$	1	1
Premiss 2	$\overline{p}$	-1	0
Conclusion	q	0	1

If a third dimension is added to the vector space in addition to the directions p and q, we can represent propositions and arguments using a third letter r, taking their place in a three-dimensional Boolean cube, and so on for any number of propositional variables.

The Hypothetical Syllogism (Figure 12) is an example of a valid argument-form which uses three variables.

The displacement matrices and their sum are:

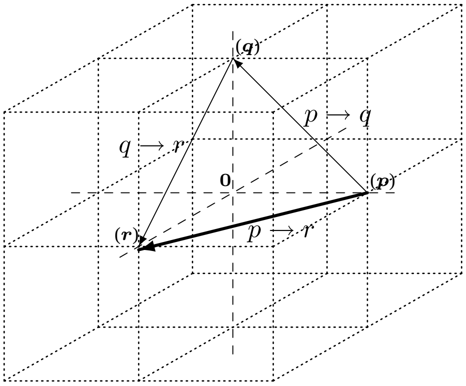

FIGURE 12. Hypothetical syllogism

		p	q	r
Premiss 1	$p \to q$	-1	1	0
Premiss 2	$q \to r$	0	-1	1
Conclusion	$p \to r$	-1	0	1

Obviously there is no finite limit to the number of possible dimensions for the vector logic space. Of course drawing perspicuous diagrams will become more difficult as the number of dimensions increases. However, this is a limitation only on the graphical representation of the vectors. The same limitation applies to any theory involving more than three dimensions. As with numerical vectors, this limitation does not prevent the use of vectors in any number of dimensions. As an example we take next Constructive Dilemma. The displacement matrices and sum are:

		p	q	r	s
Premiss 1	$p \to q$	-1	1	0	0
Premiss 2	$r \to s$	0	0	-1	1
Premiss 3	$p \vee r$	1	0	1	0
Conclusion	$q \vee s$	0	1	0	1

The displacement matrices show that the conclusion is a vector sum of the three premisses. All that remains to be shown is that the three premisses form a chain of minimal free vectors with only 1-dimensional joints. The key is finding translations of the canonical vectors that form the right sort of chain.[10] One way to do this is to take the premisses as expressed by the vectors $(\boldsymbol{p}) \rightarrowtail (\boldsymbol{q})$, $(\overline{\boldsymbol{s}}) \rightarrowtail (\overline{\boldsymbol{r}})$, and $(\overline{\boldsymbol{r}}) \rightarrowtail (\boldsymbol{p})$. Adding these vectors yields $(\overline{\boldsymbol{s}}) \rightarrowtail \boldsymbol{q}$. This can be translated to $\mathbf{0} \rightarrowtail (\boldsymbol{q}, \boldsymbol{s})$ which expresses the conclusion $q \vee s$.

Take next Double Negation, or $\overline{\overline{p}} \leftrightarrow p$. It is the *sliding* of the negative vector through the origin. If we take either of the vectors expressing $\overline{p}$ and slide it through $\boldsymbol{O}$, it turns into a vector expressing p. It seems interesting to represent the difference between the negated proposition and the activity of negating it as the difference between the position of the vector in the negative area of the space and the operation of sliding it through the origin.

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[10] Finding such a chain is the analog in our system of finding a proof in systems of propositional logic. Accordingly it is not always obvious how to find the chain. While some heuristics are available (e.g. translate as many vectors as possible so that they begin and end on an axis) we have not yet found an algorithm which will yield an appropriate chain in all cases. Possession of such an algorithm would, of course, quickly yield a proof of completeness.

Simplification and Addition are the inferential analogues of the rule of Weakening. The rules for principles of tautology (the equivalence of p to pp and to $p \vee p$), like association and commutativity, are automatic within the vector system. We recall that pq is represented as $(\boldsymbol{p}, \boldsymbol{q}) \rightarrowtail \mathbf{0}$. Conversion yields the equivalent vector system, which consists of the vectors $\boldsymbol{O} \rightarrowtail (\boldsymbol{p})$ and $\boldsymbol{O} \rightarrowtail (\boldsymbol{q})$. In the light of this, how should we represent pp? The two conjuncts must be represented like any other ANF conjunct: a vector from $\boldsymbol{O}$ to $(\boldsymbol{p})$ for the first conjunct, and a *second* vector from $\boldsymbol{O}$ to $(\boldsymbol{p})$ for the second conjunct. In graphing the first, we have also, on inspection, graphed the second. The same applies, with the vectors reversed, for $p \vee p$. This is represented as a CNF vector $(\boldsymbol{p}) \rightarrowtail \boldsymbol{O}$, but also as a *pair* of bound vectors: $(\boldsymbol{p}) \rightarrowtail \boldsymbol{O}$, and another instance of $(\boldsymbol{p}) \rightarrowtail \boldsymbol{O}$.[11]

The principles of Distribution also involve ANF/CNF conversion. This is to be expected, as the two Distribution equivalences are equivalences between an ANF and a CNF. Take the schema $pq \vee pr$ in ANF form, which is represented as a system of two bound vectors $(\boldsymbol{p}, \boldsymbol{q}) \rightarrowtail \boldsymbol{O}$ and $(\boldsymbol{p}, \boldsymbol{r}) \rightarrowtail \boldsymbol{O}$. We then convert both vectors to CNF systems (Figure 13).

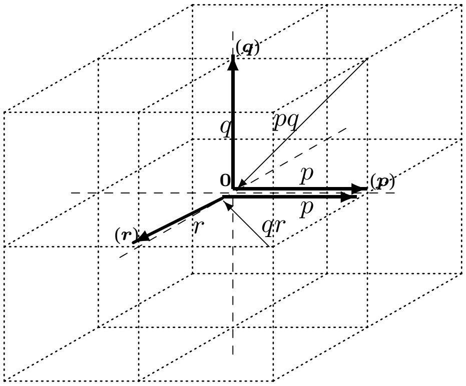

FIGURE 13. ANF $pq \vee pr$ conversion to CNF

Note that each conversion yields an instance of $\mathbf{0} \rightarrowtail (\boldsymbol{p})$, so we have two systems each with an instance of that vector. In order to reach a conclusion we must modify the two systems so that they are indistinguishable from one another, and thus become a single system. The two instances of $\mathbf{0} \rightarrowtail (\boldsymbol{p})$ collapse into one, and we can weaken $\mathbf{0} \rightarrowtail (\boldsymbol{q})$ and $\mathbf{0} \rightarrowtail (\boldsymbol{r})$ to get $\mathbf{0} \rightarrowtail (\boldsymbol{q}, \boldsymbol{r})$. The resulting vector system expresses $p(q \vee r)$ (Figure 14).

A dual pattern of transformations takes us from $(p \vee q)(p \vee r)$ to an articulated CNF equivalent, which is converted into the articulated ANF schema, $p \vee qr$.

5 Conclusion

The multidimensional logic space which we have described is a non-standard Boolean cube, with the key difference made by the presence of the origin $\boldsymbol{O}$, and the related fact that the cube consists not just of zeros and ones, but of minus-ones as well. As a result we are able to represent the *directedness* of lines within the cube, and therefore *arguments* within the space. Every vector sum, properly restricted, is a valid argument.

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[11] We are assuming here a form of the principle of the identity of indiscernibles with respect to vectors within a system - an analogue of the axiom of extensionality for vectors. However we see no reason why the system might not be modified to allow for brute facts about the identity of vectors. Such a change might be motivated, for example, by a desire to represent linear logic in a vector system. Of course this probably would entail further changes to the system.

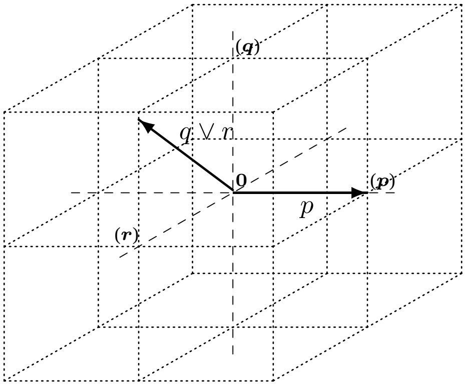

FIGURE 14. CNF $p(q \vee r)$

We have given vector logic examples of several rules of argument. While these examples are not a proof of completeness, we hope that they suggest something of the power of the vector logic approach. By harnessing the geometry of the vector space certain logical principles become trivial which would otherwise have to be proved. For example, it is an attractive feature of the vector representation that it is impossible for the symmetries of commutativity and association *not* to hold, whereas in systems of natural deduction and axiomatic systems these equivalences must be explicitly stated and justified (*e.g.* $[p \vee (q \vee r)] \leftrightarrow [(p \vee q) \vee r]$).[12] Thus we are able to reap considerable logical power from a very simple rule set.

It is clear that in order to fully validate the system of graphical vector logic described above, a proof of completeness must be given. We hope to offer one in a subsequent paper.

Acknowledgements

We wish to thank Russell Wahl, Lei Qian, and an anonymous referee for many helpful comments.

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[12] Diagrammatic systems often yield this sort of advantage. A clear discussion of the phenomenon is set forth in [11]. Interestingly, the advantages seem to hold in vector logic even when the number of dimensions makes drawing an actual diagram difficult or impossible. However, a full discussion of this fact is beyond the scope of this paper.

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Received 16 February 2004